P.G. sem-2nd
paper-VI, unit-1st-
cross-Ratio.

Cross-Ratio. The cross-ratio of the four points z_1, z_2, z_3, z_4 taken in order is defined to be equal to

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

This form is most easy to write down starting with z_1, the four elements are

$z_1-z_2,\ z_2-z_3,\ z_3-z_4,\ z_4-z_1$ in Cyclic order. The first one is put in numerator, the second in the denominator, the third in the numerator and the fourth in the denominator.

Theorem 3. The cross-ratio of four points is invariant under a bilinear transformation. IMP

Proof. Let w_1, w_2, w_3, w_4 be the images of four distinct points z_1, z_2, z_3, z_4 under a bilinear transformation

$$w = \frac{az+b}{cz+d} \quad (ad-bc \neq 0)$$

Then we shall prove that $\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

We have $w_1-w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$

$$= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

Similarly

$$w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$$

$$w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

and
$$w_4 - w_1 = \frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)}$$

$$\therefore \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \left\{ \quad \right\}$$

$$= \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

—— x ——

Problems ① Find the bilinear transformation which maps the points $z_1=1$, $z_2=-i$, $z_3=-1$ onto the points $w_1=0$, $w_2=2$ and $w_3=-i$ respectively.

Solⁿ The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{w+i}{2w} = \frac{(z-2)(1+i)}{2(z-1)} \Rightarrow \frac{i}{w} = \frac{z-2+zi-2i}{2(z-1)}$$

(3)

Substituting the given values we get

or, $\frac{i}{\omega} = \frac{-z + zi - 2i}{2(z-1)} = \frac{i^2 z + zi - 2i}{2(z-1)}$

$\omega = \frac{2(z-1)}{(1+i)z - 2}$